# 40% of a conjecture of Füredi

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In this talk, I will report on joint work with Timothy Hsu, Mark Logan, and Christopher Towse on a fifteen year old conjecture of Füredi on chain partitions of a Boolean lattice.

Consider all 8 subsets of $\{1, 2, 3\}$ and organize them into the following "chains":

$$\begin{array}{rcl} \emptyset \subseteq \{1\} & \subseteq & \{1,2\} \subseteq \{1,2,3\} \\ \{2\} & \subseteq & \{2,3\} \\ \{3\} & \subseteq & \{1,3\} \end{array}$$

You have now partitioned these subsets into chains with sizes 4, 2 & 2. But could you partition the 1024 subsets of $\{1, \ldots, 10\}$ into 16 chains of size 5 and 236 chains of size 4?

Let $[n] = \{1, \ldots, n\}$ and let $\mathbf{2}^{[n]}$ denote the poset of subsets of $[n]$ ordered by inclusion. A collection of subsets $A_0 \subset \ldots \subset A_k$ of $[n]$ is called a chain of size $k+1$ in $\mathbf{2}^{[n]}$. In this talk we discuss a construction that partitions $\mathbf{2}^{[n]}$ into a collection of chains such that the size of the shortest chain is approximately $\frac{1}{2}\sqrt{n}$ and the size of the longest chain is roughly $\sqrt{n \log n}$.

This result is motivated by conjectures of Griggs and Füredi on chain partitions of $\mathbf{2}^{[n]}$. Let $\mu = (\mu_1 \geq \ldots \geq \mu_\ell)$ be a partition of $2^n$ into positive parts. In 1988, Griggs made a conjecture that gives a necessary and sufficient condition for the existence of a partition of $\mathbf{2}^{[n]}$ into chains of sizes $\mu_1, \ldots, \mu_\ell$. Earlier, Füredi had considered a special case of the Griggs conjecture and asked if $\mathbf{2}^{[n]}$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains such that the size of every chain is one of two consecutive integers. In such a (hypothetical) construction, the size of the shortest chain will be approximately $\sqrt{\pi/2}\sqrt{n}$. In contrast, our construction gives a chain partition with the correct number of chains, and with minimal chain size roughly $\frac{1}{2}\sqrt{n}$.